Figure 1 shows the commonly accepted explanation of damping provided by viscoelastic coatings. The damping effectiveness is related to the vibratory energy losses developed in the coating material as a result of shear stresses produced by deflection of the damped structure. Kahawa and Krokstad[3] derived an expression for this type of damping effectiveness, $\beta$.

$$\beta_{\mathrm{BEND}} = 3g_E r_E r_h(1 + 2r_h + 4/3r_h^2) \quad (1)$$

where $g_E$ is the loss factor of viscoelastic material, $r_E$ is the ratio of coating bending modulus to that of structure (includes consideration of Poisson's ratios), and $r_h$ is the ratio of coating thickness to thickness of structure.

This equation shows damping to be proportional to the loss factor peculiar to the particular viscoelastic material, the ratio of the coating modulus to the modulus of the structure, and the ratio of the coating thickness to the thickness of the structure. As indicated in Fig. 1, the damping effectivity is frequently enhanced by the addition of a rigid septum over the outer surface of the coating which then forces additional shear losses in the viscoelastic material.

Figure 2a shows the mechanism by which we have accounted for the unexpected damping properties of flame-sprayed *ceramic* materials. The ceramic category includes the refractory materials, such as magnesium zirconate, ROKIDE,** alumina, and zirconia. When sprayed, these materials leave the gun in a molten state, but do not fuse to the surface on which they are sprayed. The bonding process is a mechanical one in which the ceramic flows around the macroscopic surface irregularities with no metallurgical interactions. The bonding of individual ceramic particles to each other is a combination of fusion and mechanical bonding, since some of the particles have cooled to the solid state prior to impact on the surface. Ceramic materials have no viscoelastic properties; their damping stems from an entirely different phenomenological source. The dissipation of vibratory energy is, perhaps, derived from the friction forces generated at the mechanically bonded interfaces between the coating and the damped structure, as well as at interfaces with adjacent coating fused areas.

As shown in Fig. 2b, the damping mechanism for flame-sprayed *metallic* materials, such as molybdenum and copper-nickel, is very similar. However, in the case of these materials, some fusion with the surface material does occur, and the bond mechanism is partly fusion and partly mechanical. Figure 3 shows a means for increasing the damping effectiveness of flame-sprayed coatings. The principle is the same as that used to increase the damping of viscoelastic materials, but in this case we are developing greater friction losses in the mechanically bonded interfaces.

To date we have seen no explanation in the literature for the damping provided by flame-sprayed materials and no recognition of the existence of such damping properties. While Lull's patent[4] (which covers a damping coating treatment consisting of a low modulus plating overlaid with a high modulus plating) may seem to bear some similarity, the basic principles involved are entirely different. Lull[4] attempted to develop a viscoelastic-like loss through enhanced shear stresses in the lower modulus coating. Neither that treatment nor the conventional viscoelastic treatments introduces the opportunity for the internal mechanical friction losses believed to be the damping source in this application of flame-sprayed materials.

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## Conclusions

1) A new damping mechanism is believed to be revealed which involves mechanical friction within flame-spray coating materials, probably unique to the method of application and associated bonding mechanism represented by the flame-spray technique.

2) The use of layered coatings of different rigidity (modulus) appears to enhance this internal friction damping phenomena.

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# Structure of Betz Vortex Cores

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THIS Note is concerned with the radial distribution of the tangential speed $v_T$ in a rolled-up vortex behind a lifting wing. We are specifically concerned with the situation when roll-up is just completed; the flow has become essentially two-dimensional (the presence of the wing is no longer felt acutely), but vortex decay has not yet set in measurably. A key question is: to what extent is the vortex structure at this stage defined by the roll-up process as a potential flow mechanism, and how important, by contrast, is the role of viscosity (of turbulent shear)? Assuming different answers to this question, a variety of vortex models have been proposed. Recent observations (e.g., trailing vortices may persist for long times; the viscous core is much smaller than had been predicted) seem to indicate that the potential flow mechanism is of overriding importance. Indeed, as was pointed out first by Donaldson,[1] a "forgotten" early model that disregards viscosity entirely seems to fit experiments much better than any one of the later models. This has been well confirmed by the later experiments of Mason and Marchman[2] and of Brown.[3]

The "forgotten" model is that of Betz[4] (1932). Let us denote by "core" that part of the vortex where it differs from a Rankine vortex (i.e., where not $v_T \sim 1/r$). This core consists of a small inner core where the flow is so dominated by viscosity that, roughly, $v_T \sim r$, and a much larger outer core. The Betz argument is concerned only with the outer core; it shows that, by accounting for the laws of conservation of momentum in potential flow, one

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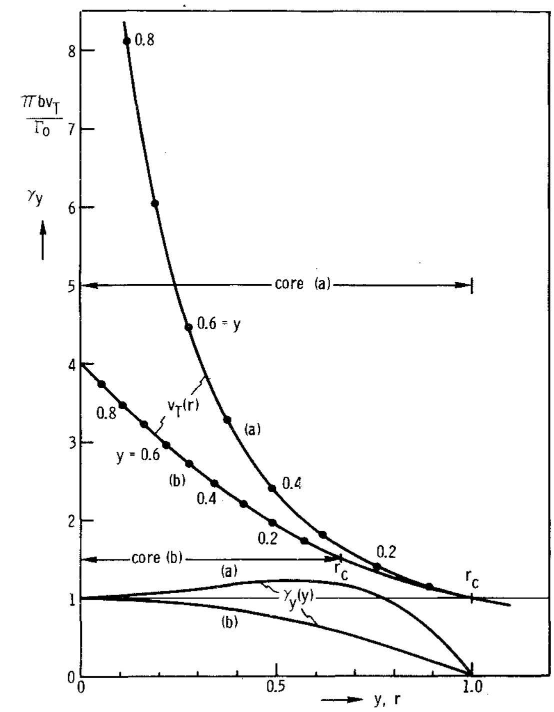


**Fig. 1 Comparison of two Betz vortices.**

arrives at an unambiguous description of the distribution of vorticity in this outer core.

Betz[4] deals specifically only with the case of an elliptic lift distribution, and one is left with the impression that other lift distributions would require somewhat complicated analyses. This impression is not dispelled by the recent papers[2,3] which have dealt with other lift distributions. The purpose of the present Note is to communicate a formula which is derived using the Betz argument, which is valid for more general lift distributions, and which is simple and easy to evaluate. Using this formula, one can (i) investigate the effect, on the core structure, of varying the wing lift distribution; (ii) distinguish in an experimental result, potential flow effects and viscosity effects; and (iii) look for a criterion that indicates whether a given lift distribution will produce single or multiple vortex pairs.

The Betz argument is not meant to be exact or complete. It starts with the usual simplifying assumption that one can discuss the individual vortex without regard to the nonuniformity of the induced speed from the necessarily present opposite vortex. Let the spanwise distribution of circulation over the wing be $\gamma_y(y)\Gamma_0$ (thus, $\gamma_y(0) = 1$) and let the radial distribution of circulation in the vortex be $\gamma_r(r)\Gamma_0$ (thus, $\gamma_r(r) = 1$ outside the core). As the wing-shed vorticity rolls up, from $y = 1$ toward $y = 0$, a single vortex builds up, with $r$ increasing from $r = 0$ to $r = r_c$. Relate $r$ and $y$ by the requirement

$$\gamma_y(y) = \gamma_r(r) \tag{1}$$

This establishes a function $r = r(y)$ which obeys the equation

$$r^2(y) = I'(y)/\gamma'_y(y) \tag{2}$$

Here $I(y)$ measures the inertia momentum of the vorticity which is shed between the tip and the point $y$. The prime (′) denotes differentiation.

Betz[4] assumed an elliptic lift distribution, that is, $\gamma_y = (1 - y^2)^{1/2}$. Without making this specialization, one obtains from Eq. (2), after some manipulation, the unexpectedly simple general formula

$$r(y) = \frac{1}{\gamma_y(y)} \int_y^1 \gamma_y(\eta)d\eta \tag{3}$$

For the elliptic lift distribution, Eq. (3) yields

$$2r(y) = (\varphi/\sin\varphi) - \cos\varphi \tag{3a}$$

where $\sin\varphi = \gamma_y$. This simple result can be seen to agree with the more lengthy result of Betz Eq. (22).[4]

Equation (3) leads directly to a very interesting conclusion. The vortex center is situated at

$$y_c = \int_0^1 \gamma_y(\eta)d\eta \tag{4}$$

The core radius is $r_c = r(0)$. Comparing Eqs. (3) and (4), one sees that $r_c = y_c$: the cores of the vortex pair just touch each other at the plane of symmetry, $y = 0$. That this occurs for the elliptic lift distribution was already noted by Betz; it is now seen to be generally true. The two inner cores are enclosed by a combined outer core, which has a combined width of between about 4/3 and 2 wing spans. This outer core should have an important effect on vortex stability; to this point we will return later.

Before applying Eq. (3) to some illustrative examples, we note again that the Betz argument is an approximation (which, however, has proven quite adequate for its intended purpose), and note also, for completeness, that the axial speed must be adjusted in order to fulfill the Bernoulli equation. This could be done quite readily, using Batchelor[5] [Eq. (2.8)], if one knew how the speed defect in the boundary layer shed by the wing is distributed radially in the vortex. Brown[3] follows Batchelor[5] (who followed Mangler and Smith) in assuming that the boundary layer rolls up with the vortex sheet. In contrast, the Manson and Marchman[2] experiments would seem to indicate (Fig. 15) that the b.l., having lost its momentum, tends to gravitate into the pressure sink at the vortex center. This, in turn, would imply that the outer core is pushed outward (which also can be seen in the experiments). However, it is not the purpose of this note to pursue this type of consideration.

We turn next to item (i), the effect of $\gamma_y(y)$ on the structure of the outer core. We do this keeping item (iii) in mind. From experiments, it is known that multiple vortex pairs can form behind the wing if the lift distribution has a pronounced local dip. Under such conditions, each half wing sheds vorticity of both signs, and vortex roll-up may start at both the tip and the dip.

The Betz argument assumes that the vortex roll-up proceeds monotonically from the tip to the wing center. One could try to modify this analysis, but, in order to do so, some criterion concerning when, where and how a separate roll-up occurs is needed. Such a criterion must come from experiments. In the present Note, we discuss only what can be learned from Eq. (3).

Because of symmetry, we have, at the wing center, $\gamma_y'(0) = 0$ and, from Eq. (2), the indeterminate result $r^2 = (0/0)$. This apparent difficulty resolves itself, however, because $y = 0$ is the end of the $y$-range under consideration.

On the other hand, if the lift distribution has a dip, at least one point $y \neq 0$ exists where the shed vorticity changes sign. Here, again, Eq. (2) yields (0/0), which might be taken as a "warning signal." Therefore, in deriving Eq. (3), we assumed that no dip occurs. However, the end result, Eq. (3), no longer contains a derivative, and inspection shows that this equation yields smooth and monotonic functions $r(y)$ even if there is a moderate dip. Indeed, since no vorticity is shed at a point where $\gamma_y'(y) = 0$, one may argue that the core at the corresponding point $r(y)$ must approximate a Rankine vortex. This, in fact, is implied in Eq. (3), where there is no longer a "warning signal."

To illustrate, Fig. 1 presents two numerical results derived from Eq. (3). They correspond to the two distributions†

$$\gamma_y(y) = 1 + 1.5y^2 - 2.5y^4 \qquad (5a)$$

$$\gamma_y(y) = 1 - y^2 \qquad (5b)$$

which are shown as distributions (a) and (b) in Fig. 1. Distribution (a) has a slight central dip and is chosen such that $r_c = y_c = 1$; the separation distance between the two trailing vortices just equals the wing span. With a deeper dip, the separation distance would be even larger. In contrast, for distribution (b), one finds $r_c = y_c = 2/3$, smaller than for the elliptic lift distribution.[3]

The induced drag of both (a) and (b) obeys the formula

$$C_{Di} = (9/8)C_L{}^2/\pi A \qquad (6)$$

The factor (9/8) in Eq. (6) is replaced by 1 in the case of minimum induced drag (the elliptic lift distribution).

For each distribution $\gamma_y$, Fig. 1 shows the distribution $v_T(r)$ of the tangential speed in the core.‡ On the $v_T$-curves, the points y on the wing are noted where the local changes of $\gamma_r(r)$ in the core are shed. The structure of the vortex core depends considerably upon the lift distribution. Distribution (a), having the larger total lift, has to have the larger energy content according to Eq. (6). This is reflected in the larger core speeds $v_T$. Outside their cores, the two vortices are identical with the same Rankine vortex.

Distribution (a) has a horizontal tangent at about $y = 0.55$. As discussed above, this does not produce any irregularity in the corresponding $v_T$ curve. This sample result suggests that the mere occurrence of a zero in $\gamma_y'$ should not be sufficient to invalidate Eq. (3). On the other hand, if the dip is so deep that the lift changes sign, then, according to Eq. (3), $r(y)$ goes to infinity and changes sign. This analytical result is physically unacceptable. Before this happens, there is a range of uncertainty. It must be left to the experiment to establish the criterion to be used.

We close with a remark on stability analyses. In the experiments of Bilanin and Widnall,[6] the spanwise lift distribution was varied periodically by means of flaps, somewhat in the manner of Fig. 1, but such that the total lift remained about constant (see Fig. 3 of Ref. 6). According to Eq. (3), the relation between $y$ and $r$ remained about invariant in the vicinity of the tip, which corresponds to the region near the inner core. Consequently, since the circulation shed near the tips was strongest at the times when the outer flaps were down, the circulation next to the inner core (the essential agent which produces self-induction) was *strongest* at the corresponding positions along the trailing vortices. At the same positions, the Rankine vortices were at their *weakest*. In the analysis,[6] no allowance was made for the outer cores; the Rankine vortices were assumed to reach to the inner cores. One sees from this example that Eq. (3) might open the way to a more realistic stability analysis.

†Neither distribution fulfills the tip condition $\gamma_y'(1) = -\infty$ (which is immaterial for this illustration).

‡From Eq. (5b), Eq. (3) yields $r = (2 - y - y^2)/3(1 + y)$. The reader may compare this result with the much more complicated form of Brown[3] Eq. (9) to illustrate the simplification which arises from Eq. (3).


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# Design of Crashworthy Aircraft Cabins Based on Dynamic Buckling


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### Nomenclature

- $A$ = stiffener cross-sectional area (in.$^2$)
- $E$ = Young's modulus (psi)
- $\bar{F}_{cr}$ = nondimensional buckling load, $\bar{F}_{cr} = \hat{N}_{cr}R/Eh^2$
- $h$ = unstiffened shell wall thickness (in.)
- $I_0$ = moment of inertia of stiffener about shell middle surface (in.$^4$)
- $J$ = polar moment of inertia of stiffener about its centroid (in.$^4$)
- $l, L$ = ring center-to-center spacing and shell length (in.)
- $M, n$ = number of stringers and number of circumferential waves
- $\hat{N}_{cr}$ = externally applied axial-load resultant for dynamic buckling (positive in comp)
- $r, s$ = subscript referring to ring and stringer quantities
- $R$ = radius of cylinder middle surface (in.)
- $\bar{z}$ = distance from centroid of stiffener to shell middle surface (in.)

### Introduction

ANALYTICAL or experimental investigations of the structural integrity of light aircraft cabins under crash impact conditions have been lacking. Light aircraft have design loadings much lower than those of military or transport design and thus have fewer stiffeners, i.e., much wider stiffener spacing. Therefore, the popular analytical approach of considering stiffeners to be "smeared" over the shell surface[1] is not valid for light aircraft and one should account for the discreteness of the stiffeners, as has


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